

# SEARCHING FOR THE MAXIMAL VALENCE OF HARMONIC POLYNOMIALS: A NEW EXAMPLE

SEUNG-YEOP LEE, ANDRES SAEZ

ABSTRACT. We find a new lower bound for the maximal number of zeros to harmonic polynomials,  $p(z) + \overline{q(z)}$ , when  $\deg p = n$  and  $\deg q = n - 2$ .

## 1. INTRODUCTION AND RESULT

Given two polynomials  $p(z)$  and  $q(z)$  of degrees  $n$  and  $m$  respectively, the maximal number of roots (i.e. maximal valence) of the *harmonic polynomial*,  $p(z) + \overline{q(z)}$ , is not known [11] except for a few cases (e.g. when  $m = n - 1$  [13, 14] and when  $m = 1$  [9, 10]). See also [2, 5, 6, 7, 8, 12]. Recently there have been several results [1, 3, 4] on the *lower bounds* of the maximal valence, see Table 1.

TABLE 1. The known maximal valence of  $p + \bar{q}$

$(\deg p, \deg q)$	$(n, m)$	$(n, n - 1)$	$(n, n - 3)$	$(n, 1)$
maximal valence	$\geq m^2 + m + n$	$n^2$	$\geq n^2 - 3n + \mathcal{O}(1)$	$3n - 2$

In this paper we suggest a new lower bound of the maximal valence when  $(\deg p, \deg q) = (n, n - 2)$ , by studying specific harmonic polynomials defined below.

Given a positive integer  $n$  let us define two polynomials,  $p(z) = S(z) + T(z)$  and  $q(z) = S(z) - T(z)$ , where

$$(1) \quad S(z) = iz^n, \quad T(z) = i(z + 1)^{n-1}(z - (n - 1)).$$

It follows that  $\deg p = n$  and  $\deg q = n - 2$ . Since the maximal valence for  $(n, m) = (3, 1)$  is known (see the above table), we only consider  $n \geq 4$  in this paper.

**Theorem.** *Given  $n \geq 4$ , let the polynomials  $p$  and  $q$  be given as above. Let  $k_{\max}(n)$  be defined by*

$$k_{\max}(n) = \max_{1 \leq k \leq n/2} \left\{ k : (n - 2) \cot \left( \frac{2k - 1}{2n - 4} \pi \right) - n \cot \left( \frac{\pi k}{n} \right) > 0 \right\}.$$

*Then the total number of zeros, counting the multiplicity, of  $p(z) + \overline{q(z)}$  is given by*

$$n^2 - 2n + 2 + 4k_{\max}(n).$$

*The asymptotic behavior of  $k_{\max}(n)$  as  $n \rightarrow \infty$  is given by*

$$k_{\max}(n) = \left( \frac{1}{4} - \frac{X}{2\pi} \right) n + \mathcal{O}(1) \approx 0.13237n + \mathcal{O}(1)$$

*where  $X \approx 0.73908513321516$  is the unique solution to the equation  $X = \cos X$ .*

TABLE 2. The number of zeros,  $n^2 - 2n + 2 + 4k_{\max}(n)$ , of  $p + \bar{q}$  for small  $n$ 's

$n$	Number of zeros	$n$	Number of zeros
4	10	20	370
5	17	21	409
6	26	22	450
7	37	23	497
8	54	24	542
9	69	25	589
10	86	26	638
11	105	27	689
12	126	28	742
13	149	29	797
14	174	30	854
15	201	31	917
16	234	32	978
17	265	33	1041
18	298	34	1106
19	333	35	1173

**Remark 1.** For general  $n$  and  $m$ , there exists a conjecture by Wilmschurst [13] on the largest valence of the harmonic polynomials. Though the conjecture has been disproved [4, 3] for a number of cases, it has not been checked for many other cases including the case considered in this paper. Our theorem says that the maximal valence is greater at least by

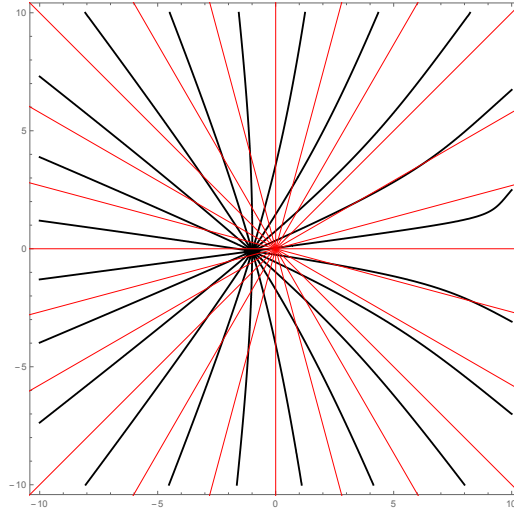
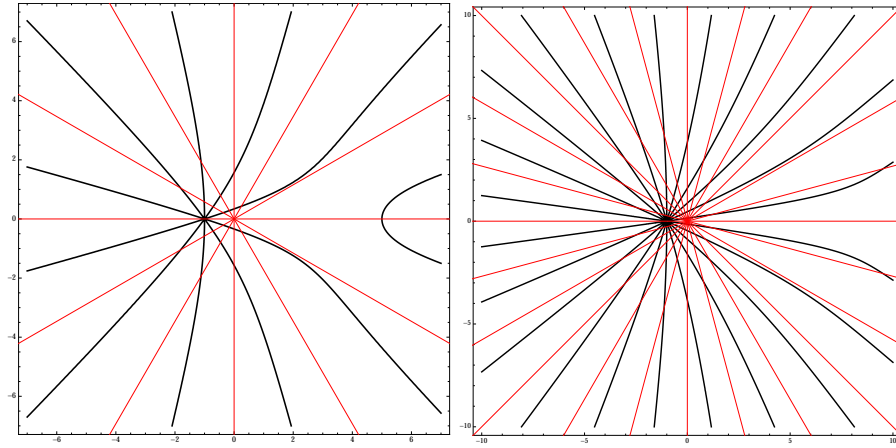
$$4k_{\max}(n) - 2 \approx 0.52948n + \mathcal{O}(1) \quad \text{as } n \rightarrow \infty$$

than the conjectured value of  $n^2 - 2n + 4$ . Our theorem also improves upon the more recent conjecture by the authors that suggests  $n^2 - 3n/2 + \mathcal{O}(1)$  for the asymptotic maximal valence as  $n$  grows to the infinity. In fact, the current project is motivated by the latter conjecture.

**Remark 2.** The specific harmonic polynomials that we consider in this paper are not new. The same polynomials appeared in [4]. However, instead of obtaining a *lower bound* on the number of roots, here we obtain the *exact* number of roots for the given polynomials. (If one naively applies the method in [4] one would only get  $n^2 - 2n + 2$  as a lower bound.) It is curious whether the similar analysis (i.e. exact counting) can be done for the case of  $m = n - 3$  that was considered in [4].

**Remark 3.** We note that our valence,  $n^2 - 2n + 2 + 4k_{\max}(n)$ , is not be the maximal valence. In fact, for some  $n$ 's, we could find harmonic polynomials with higher valence. The example shown in Figure 1 is generated by defining  $p(z) = S(z) + T(z)$  and  $q(z) = S(z) - T(z)$ , where  $S(z) = iz^{12}$  and  $T(z) = i(z + e^{\frac{i}{10}})^{11}(z - 11e^{\frac{i}{10}})$ . The number of zeros is 128, two more than 126. We conjecture that the maximal valence is either  $n^2 - 2n + 2 + 4k_{\max}(n)$  or  $n^2 - 2n + 4 + 4k_{\max}(n)$  depending on  $n$ .

**Remark 4.** We conjecture that, given the degrees,  $n = \deg p$  and  $m = \deg q$ , there exists no polynomial formula in  $n$  and  $m$  that gives the maximal valence of the harmonic polynomial  $p + \bar{q}$  for all (except possibly for a *finite* number of cases)  $n$  and  $m$ . If there exists such a polynomial formula, say  $P(n, m)$ , then for  $m = n - 2$ , we must have  $P(n, n - 2) = n^2 + An + B$  for some


 FIGURE 1. The zero sets of  $\text{Im } T$  (black) and  $\text{Re } S$  (red) as defined in Remark 3.

 FIGURE 2. The zero sets of  $\text{Re } S$  (red) and  $\text{Im } T$  (black) for  $n = 6$  and  $12$  (from left)

constants  $A$  and  $B$ . The constants  $A$  and  $B$  must be integers since  $P(n, n-2)$  must be an integer for each  $n$ . Our theorem indicates that the only possibilities are either  $A = 0$  or  $A = -1$ . To match the known cases,  $P(3, 1) = 7$  and  $P(4, 2) \in \{12, 14, 16\}$ <sup>1</sup>, one must have  $A = 0$  and  $B = -2$ . And this gives  $P(n, n-2) = n^2 - 2$  which is unlikely based on the known data.

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## 2. PROOF

We assume  $n \geq 4$  throughout this section.

From (1) above, we have  $p(z) + \overline{q(z)} = 2 \operatorname{Re} S(z) + 2i \operatorname{Im} T(z)$ . Therefore, the zeros of  $p(z) + \overline{q(z)}$  are exactly the intersection points of  $\{z : \operatorname{Re} S(z) = 0\}$  and  $\{z : \operatorname{Im} T(z) = 0\}$ . In Figure 2 the former set is depicted in the red lines and the latter set in the black curves. The zero set of  $\operatorname{Re} S$  is given explicitly by the union of  $2n$  rays, i.e.,

$$\{z : \operatorname{Re} S(z) = 0\} = \bigcup_{k=-n+1}^n \{re^{i\pi k/n} : 0 \leq r < \infty\}.$$

Therefore, to find the zeros of  $p(z) + \overline{q(z)}$ , it is enough to find the number of intersections on each ray, i.e.

$$N_k := \#\{r \in (0, \infty) : \operatorname{Im} T(re^{i\pi k/n}) = 0\},$$

for each  $k = -n+1, \dots, n$ .

From the expression of  $T$  in (1), we obtain  $N_0 = 1$  and  $N_n = n-1$  (counting the degeneracy). We also obtain that  $\operatorname{Im} T(re^{-i\pi k/n}) = \operatorname{Im} T(re^{i\pi k/n})$  and, therefore,  $N_k = N_{-k}$ . As a result we only need to find  $N_k$  for  $k = 1, \dots, n-1$ .

**Lemma 1.** *For  $1 \leq k \leq n-1$ ,  $N_k$  is given by the number of zeros of  $A : (0, \pi k/n) \rightarrow \mathbb{R}$  where*

$$A(\theta) = A(\theta; k) = \tan[(n-1)\theta] + \frac{n-1}{\tan \theta} - n \cot \frac{\pi k}{n}.$$

*Proof.* Let us evaluate

$$\arg T(re^{i\pi k/n}) = (n-1)\theta(r) + \phi(r) \pmod{2\pi},$$

where we define the two (angular) variables,

$$(2) \quad \begin{aligned} \theta(r) &= \theta(r; k) = \arg(re^{i\pi k/n} + 1), \\ \phi(r) &= \phi(r; k) = \arg(re^{i\pi k/n} - (n-1)) + \frac{\pi}{2}. \end{aligned}$$

This gives

$$(3) \quad \tan \arg T(re^{i\pi k/n}) = \frac{\tan((n-1)\theta(r)) + \tan \phi(r)}{1 - \tan((n-1)\theta(r)) \tan \phi(r)}.$$

Using the identities

$$\begin{aligned} \tan \theta(r) &= \tan \arg(re^{i\pi k/n} + 1) = \frac{r \sin(\frac{\pi k}{n})}{r \cos(\frac{\pi k}{n}) + 1}, \\ \tan \phi(r) &= \tan\left(\arg(re^{i\pi k/n} - (n-1)) + \frac{\pi}{2}\right) = -\frac{r \cos(\frac{\pi k}{n}) - (n-1)}{r \sin(\frac{\pi k}{n})}, \end{aligned}$$

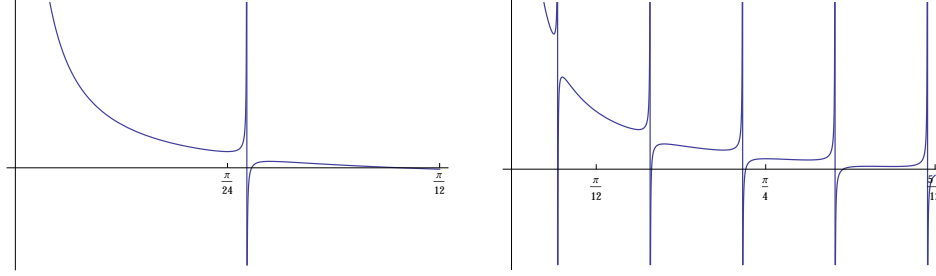
one can relate  $\phi$  and  $\theta$  by  $\tan \phi(r) = \frac{n-1}{\tan \theta(r)} - n \cot \frac{\pi k}{n}$ . Using this relation, the numerator of  $\tan \arg T(re^{i\pi k/n})$  in (3) is written as

$$\tan((n-1)\theta(r)) + \tan \phi(r) = \tan((n-1)\theta(r)) + \frac{n-1}{\tan \theta(r)} - n \cot \frac{\pi k}{n},$$

which is exactly  $A(\theta(r))$  as defined in the lemma. Since we have  $\operatorname{Im} T(z) = 0$  (note  $T(z) \neq 0$  away from the real axis) if and only if  $\tan \arg T(z) = 0$ ,  $N_k$  is given by the number of zeros of

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<sup>1</sup>For  $(n, m) = (4, 2)$  the maximal valence achieved so far is 12. The maximal valence for even  $n$  needs to be even due to the argument principle. See, for example, [1].

FIGURE 3. Plots of  $A(\theta)$  for  $n = 12$  and  $k = 1$  (left) and  $k = 5$  (right)

$A(\theta(r))$  over  $r \in (0, \infty)$ . It is simple to check that the denominator in (3) does not vanish when the numerator vanishes.

Lastly, from (2), one can see that  $\theta(r)$  (the angle from  $-1$  to a point on the straight ray parametrized by  $r$ ) increases from zero to  $\pi k/n$  monotonically as  $r$  moves from zero to  $\infty$ . ■

In the rest of the proof, we will use elementary argument (e.g. the mean value theorem and the intermediate value theorem) to count the zeros of  $A(\theta)$ . See Figure 3 for some plots of  $A$ .

One notices that  $A$  has simple poles (of negative residue) where  $\tan[(n-1)\theta]$  has poles, i.e.,

$$\theta = \frac{1/2}{n-1}\pi, \frac{3/2}{n-1}\pi, \dots, \frac{k_{\text{poles}} - 1/2}{n-1}\pi,$$

where  $k_{\text{poles}}$  is the largest integer such that  $\frac{k_{\text{poles}} - 1/2}{n-1} < \frac{k}{n}$ . We have

$$(4) \quad k_{\text{poles}} = k_{\text{poles}}(k) = \begin{cases} k & \text{when } k < n/2, \\ k-1 & \text{when } k \geq n/2. \end{cases}$$

We also get the boundary behavior of  $A$ :  $\lim_{\theta \rightarrow 0+} A(\theta) = +\infty$  and

$$(5) \quad A\left(\frac{\pi k}{n}\right) = -\tan\left(\frac{\pi k}{n}\right) - \cot\left(\frac{\pi k}{n}\right) = -\frac{2}{\sin(2\pi k/n)} = \begin{cases} \leq 0 & \text{when } k < n/2, \\ > 0 & \text{when } k \geq n/2. \end{cases}$$

(When  $k = n/2$ ,  $\lim_{\theta \rightarrow \pi k/n} A(\theta) = +\infty$ .) To find the critical points of  $A$ , we evaluate

$$A'(\theta) = (n-1) \left( \frac{1}{\cos^2[(n-1)\theta]} - \frac{1}{\sin^2 \theta} \right),$$

that becomes zero when

$$\cos[(n-1)\theta] = \pm \sin \theta \iff (n-1)\theta = \frac{\pi}{2} \pm \theta + \pi j \text{ (for } j \in \mathbb{Z}),$$

or, equivalently, when

$$\theta = \begin{cases} \frac{j-1/2}{n}\pi & \text{where } j = 1, \dots, k, \\ \frac{j-1/2}{n-2}\pi & \text{where } j = 1, \dots, k_{\text{crit}}. \end{cases}$$

We note that  $k_{\text{crit}}$  is the largest integer such that  $\frac{k_{\text{crit}}-1/2}{n-2} < \frac{k}{n}$  or, equivalently,

$$k_{\text{crit}} = k_{\text{crit}}(k) = \begin{cases} k & \text{when } k < n/4, \\ k-1 & \text{when } n/4 \leq k < 3n/4, \\ k-2 & \text{when } k \geq 3n/4. \end{cases}$$

The critical values are given by

$$(6) \quad \begin{cases} A\left(\frac{j-1/2}{n-2}\pi\right) = (n-2)\cot\left(\frac{j-1/2}{n-2}\pi\right) - n\cot\left(\frac{\pi k}{n}\right), \\ A\left(\frac{j-1/2}{n}\pi\right) = n\cot\left(\frac{j-1/2}{n}\pi\right) - n\cot\left(\frac{\pi k}{n}\right) > 0. \end{cases}$$

The last inequality is from the monotonicity of  $\cot(x)$  over  $0 < x < \pi$ .

**Lemma 2.** For  $1 \leq k \leq n-1$  we have

$$A\left(\frac{j-1/2}{n-2}\pi\right) > 0 \quad \text{for } 1 \leq j \leq \min\{k_{\text{crit}}, k-1\}.$$

*Proof.* Since  $A\left(\frac{j-1/2}{n-2}\pi\right)$  is monotonic in  $j$ , it is enough to prove that

$$A\left(\frac{k'-1/2}{n-2}\pi\right) = (n-2)\cot\left(\frac{k'-1/2}{n-2}\pi\right) - n\cot\left(\frac{\pi k}{n}\right) > 0,$$

for  $k' = \min\{k_{\text{crit}}, k-1\}$ . Using the following identity,

$$(7) \quad (n-2)\cot\theta_1 - n\cot\theta_2 = \frac{(n-1)\sin(\theta_2 - \theta_1) - \sin(\theta_1 + \theta_2)}{\sin\theta_1 \sin\theta_2},$$

the above inequality in question becomes

$$\begin{aligned} F_1(k) &:= (n-1)\sin\left(\frac{3n-4k}{2n^2-4n}\pi\right) - \sin\left(\frac{k-3/2}{n-2}\pi + \frac{\pi k}{n}\right) > 0 \text{ when } k < 3n/4, \\ F_2(k) &:= (n-1)\sin\left(\frac{5n-4k}{2n^2-4n}\pi\right) - \sin\left(\frac{k-5/2}{n-2}\pi + \frac{\pi k}{n}\right) > 0 \text{ when } k \geq 3n/4. \end{aligned}$$

We have  $F_1(k) > 0$  when

$$k > C_n := \frac{n(2n-1)}{4(n-1)}$$

because both terms in  $F_1(k)$  contribute positively (we defined  $C_n$  such that the second term of  $F_1(k)$  vanishes when  $k = C_n$ ). For  $k \leq C_n$ , since the first term in  $F_1(k)$  decreases monotonically in  $k$  and is given by (using  $\sin x > \frac{2x}{\pi}$  for  $0 < x < \pi/2$ )

$$(n-1)\sin\frac{\pi}{2(n-1)} > 1 \quad \text{when } k = C_n,$$

$F_1(k) > 0$  for all  $k \leq C_n$ .

Similarly,  $F_2(k) > 0$  for  $k \geq 3n/4$  because both terms in  $F_2(k)$  contribute positively. ■

From (6) and Lemma 2, the only possible critical point  $\theta$  such that  $A(\theta) \leq 0$  occurs at

$$\theta = \frac{k-1/2}{n-2}\pi, \quad k < \frac{n}{4}.$$

Note that this is bigger than  $\frac{k-1/2}{n-1}\pi$  which is the location of the rightmost pole of  $A$ . Since all the other critical values are positive, it follows that, between any successive poles of  $A$  (except between  $\theta = 0$  and  $\theta = \frac{\pi}{2(n-1)}$ ), there is exactly one root of  $A$ , by applying the intermediate

value theorem and the mean value theorem. Therefore, there are total  $k_{\text{poles}} - 1$  zeros of  $A$  that are located to the left of the rightmost pole. By applying the intermediate value theorem with (5), for all  $k \geq n/2$ , there is at least one root of  $A$  to the right of the rightmost pole of  $A$ . Combining with (4) there are at least  $k - 1$  roots of  $A$  for all  $1 \leq k \leq n - 1$ . The total number of zeros of  $p + \bar{q}$  is therefore at least

$$(8) \quad n + 2 \sum_{k=1}^{n-1} (k - 1) = n^2 - 2n + 2.$$

For  $k < n/2$ , again applying the intermediate value and the mean value theorem between the rightmost pole and  $k\pi/n$ , there are

$$\begin{cases} \text{no zero when } k \geq n/4 \text{ or when } k < n/4 \text{ and } A\left(\frac{k-1/2}{n-2}\pi\right) < 0, \\ \text{two zeros when } k < n/4 \text{ and } A\left(\frac{k-1/2}{n-2}\pi\right) > 0. \end{cases}$$

Note that the second inequality is exactly the condition to define  $k_{\text{max}}(n)$  in the main theorem. Using the identity (7), the inequality,  $A\left(\frac{k-1/2}{n-2}\pi\right) > 0$ , holds if and only if

$$(n-1) \sin\left(\frac{n-4k}{2n^2-4n}\pi\right) - \sin\left(\frac{k-1/2}{n-2}\pi + \frac{\pi k}{n}\right) > 0.$$

For  $k < n/4$ , the left hand side is monotonically decreasing in  $k$  and, therefore, the inequality is satisfied exactly for  $k \leq k_{\text{max}}(n)$ . Therefore we get  $4k_{\text{max}}(n)$  more roots of  $p + \bar{q}$  than (8). This proves our theorem except the statement about the asymptotic behavior.

In terms of the new parameter,  $\gamma = k/n$ , the left hand side of the above inequality becomes

$$(n-1) \sin\left(\frac{1-4\gamma}{2n-4}\pi\right) - \sin\left(\frac{n\gamma-1/2}{n-2}\pi + \gamma\pi\right) = \frac{\pi}{2} - 2\pi\gamma - \sin(2\pi\gamma) + \mathcal{O}\left(\frac{1}{n}\right),$$

as  $n \rightarrow \infty$ . The leading term in the above expression is positive when

$$\frac{k}{n} = \gamma < \frac{1}{4} - \frac{X}{2\pi}$$

where  $X \approx 0.739$  is the unique solution to the equation  $X = \cos X$ . It implies that  $\frac{k_{\text{max}}}{n} = \frac{1}{4} - \frac{X}{2\pi} + \mathcal{O}\left(\frac{1}{n}\right)$ .

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